Relaxations for Production Planning Problems with Increasing By-products

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University of Wisconsin-Madison

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One slide summary

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  - Production process involves desirable & undesirable products.

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  - New discrete time MINLP formulation.
  - MIP Approximation & Relaxation schemes.
- **Performance evaluation**

Problem Description

Production Process

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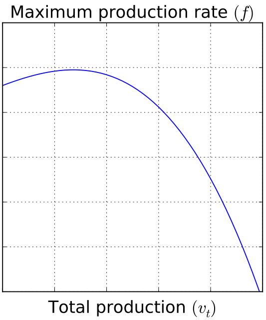
- Decisions span a planning horizon $\mathcal{T}$.
- Discrete decisions determine the start time of the production process.
- Continuous decisions determine the production profile evaluated by production functions $f(\cdot)$ and $g_p(\cdot)$.

Production functions

- Production function $f(\cdot)$ is a concave function that determines the maximum production rate as a function of total production.



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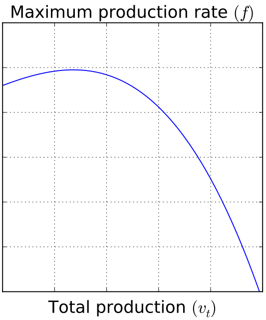

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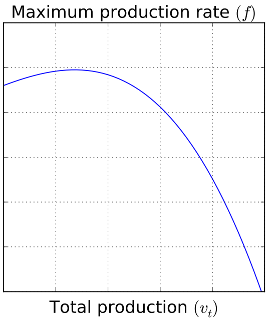

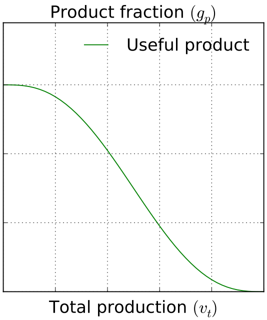

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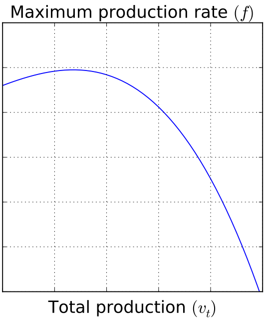

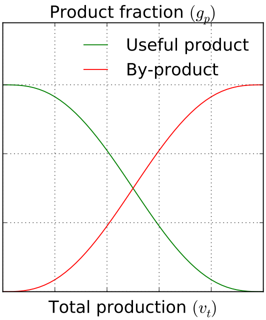

Movie time

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Continuous time formulation

Cumulative production $v(t)$ is calculated using production rate x(t)

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Production profiles are active only after the start time $z(t)$

$$v(t) = 0 \quad \forall t < z(t)$$

Discrete time MINLP formulations

Discrete time formulations

Past models have proposed a natural discretization of this continuous time model.

Continuous time formulation (F)

$$v(t) = \int_0^t x(s)\mathrm{d}s$$

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$$y_p(t) = \; x(t) \; g_p(v(t))$$

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v_t Cumulative production up to time period $t \in \mathcal{T}$.

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z_t	Facility on/off decision variable.

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Formulation F_1

How much of product p is produced up to time t?

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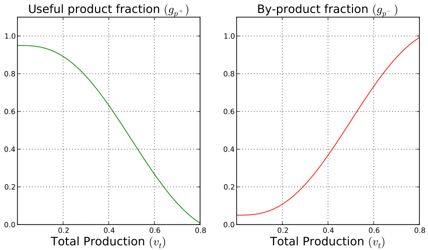

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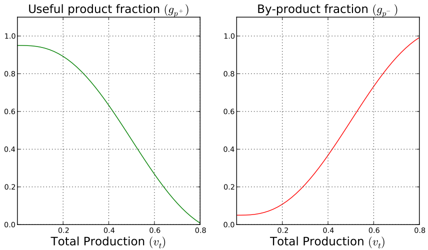

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How much product is produced up to time t?

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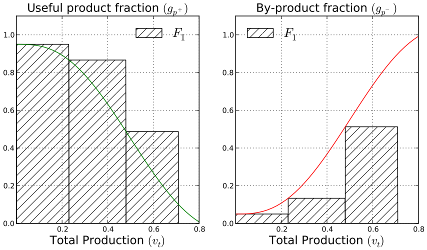

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$$= \int_0^{v_t} g_p(v)\mathrm{d}v$$

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$$\stackrel{\text{def}}{=} h_p(v_t)$$

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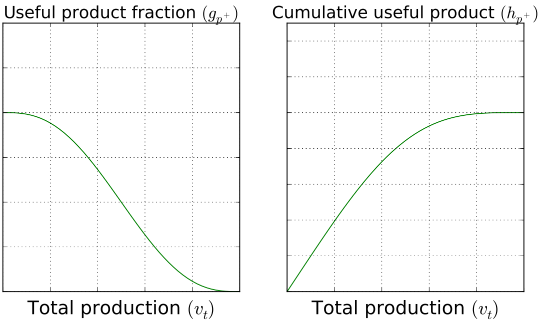
Key Idea

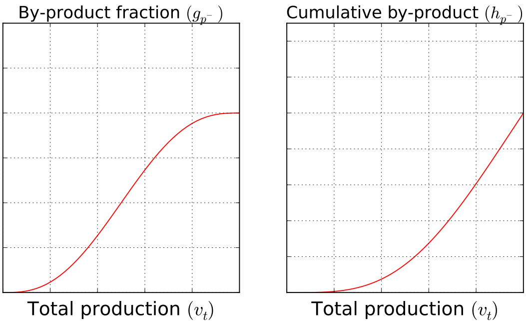
- Integral of a non-increasing function is concave .



Alternate formulation

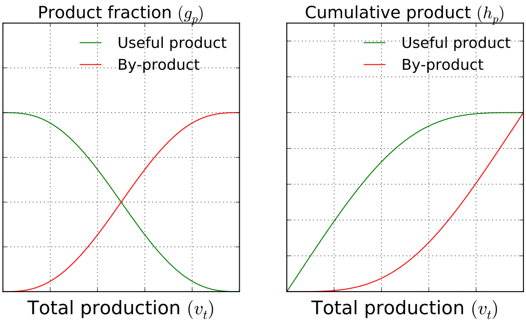
Key Idea

- Integral of a non-increasing function is concave .
- Integral of a non-decreasing function is convex.



Key Idea 

- Integral of a non-increasing function is concave .
- Integral of a non-decreasing function is convex.
- Lets deal with $h_p(\cdot)$ instead of $g_p(\cdot)$!



Comparing formulations

What have we done so far ?

F_1

$$v_t = \sum_{s=0}^{t} x_s$$

$$x_t \le \Delta_t f(v_{t-1})$$

$$y_{p,t} = x_t \, g_p(v_{t-1})$$

$$v_t \le M \, z_t$$

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Comparing formulations

What have we done so far ?

F1	F2
$v_t = \sum_{s=0}^{t} x_s$	$v_t = \sum_{s=0}^{t} x_s$
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F₂

$$v_t = \sum_{s=0}^{t} x_s$$

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F_2

$$v_t = \sum_{s=0}^{t} x_s$$

$$x_t \leq \Delta_t f(v_{t-1})$$

$$y_{p,t} = h_p(v_t) - h_p(v_{t-1})$$

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Which formulation is better?

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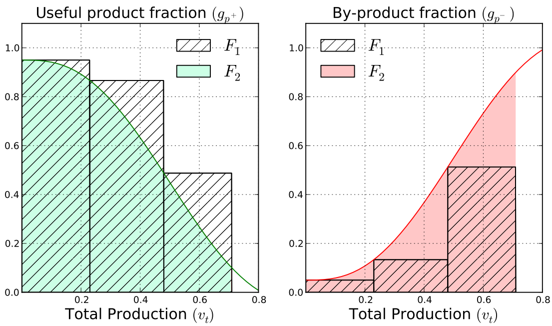
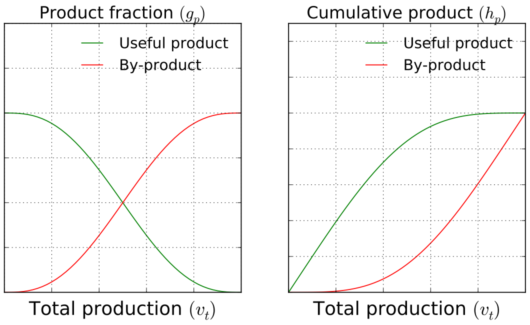
Which formulation is better?

- F_2 is a more accurate formulation of F than F_1 .



Which formulation is better?

- F_2 is a more accurate formulation of F than F_1 .
- F_2 is computationally better because it deals with convex functions while F_1 deals with bivariate functions.



MIP Approximations & Relaxations

Mixed Integer Non-Linear Programs (MINLP)

... are slow and hard!

Approximations & Relaxations

Mixed Integer Non-Linear Programs (MINLP)

... are slow and hard!

Why MINLP is like Cricket

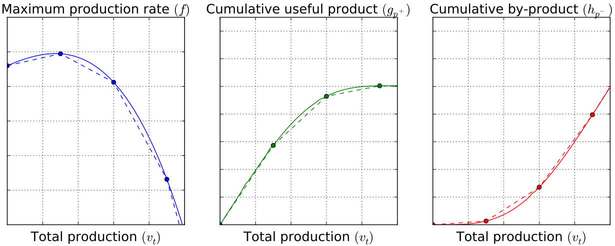
- It goes on forever.
- May not produced a result.

But...the MILP force is here

We only need to approximate or relax univariate convex and concave functions.

Piecewise Linear Approximation (PLA)

Approximate all the nonlinear production functions with piecewise linearizations.[1]



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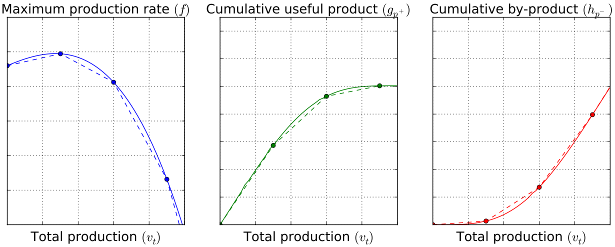

Approximations & Relaxations I

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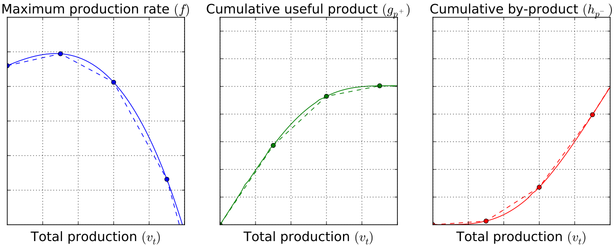

Approximations & Relaxations I

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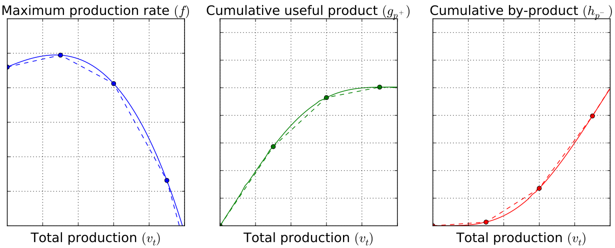
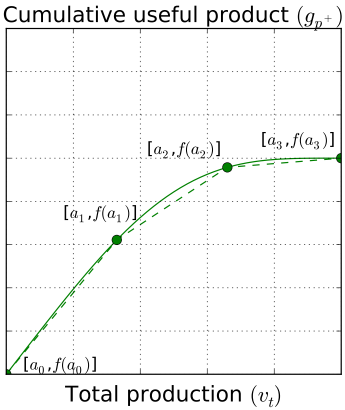


Specially ordered sets (SOS)



Approximating $f(v_t)$

$$f(v_t) \approx \sum_{o \in \mathcal{O}} \lambda_{t,o} f(a_o)$$

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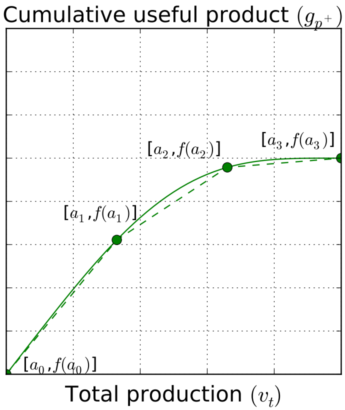

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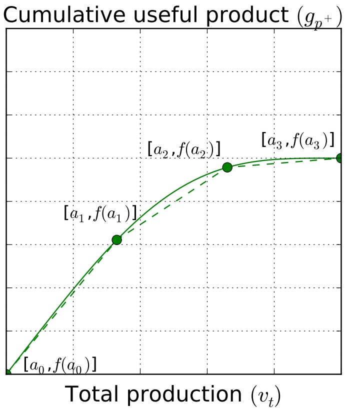

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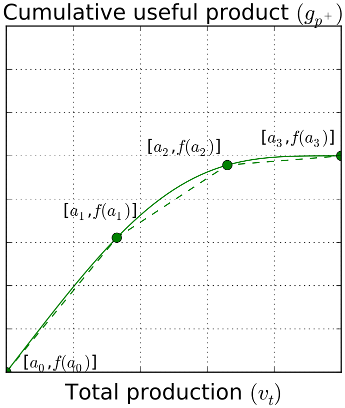

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$$v_t = \sum_{s=0}^{t} x_s$$

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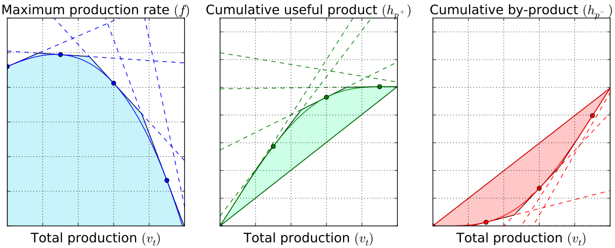
Approximations & Relaxations II

Secant Relaxation (1-SEC)

Relax all the nonlinear production functions using inner and outer approximations.[2]



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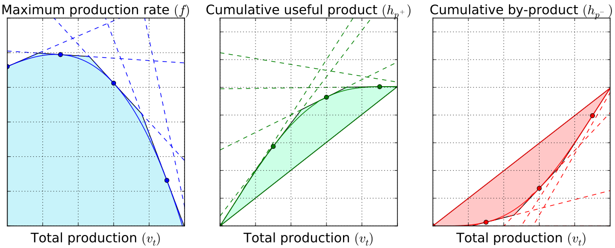

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Relax all the nonlinear production functions using inner and outer approximations.[2]

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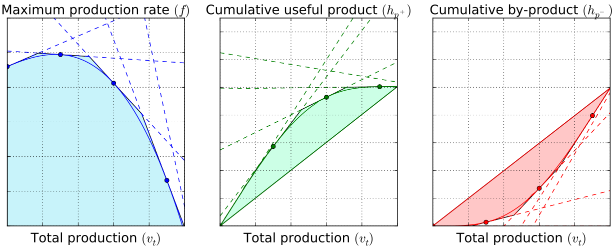
  - Does NOT introduce additional SOS2 variables.
- Cons
  - May not be ‘close’ to a feasible solution of the MINLP formulation.



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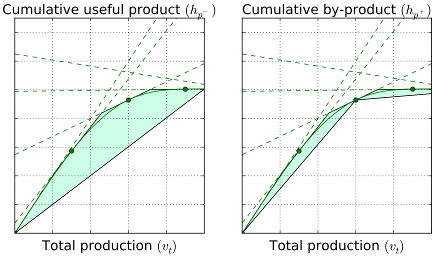
Relax all the nonlinear production functions using inner and outer approximations but use multiple secants instead of a just a single one.



Approximations & Relaxations III

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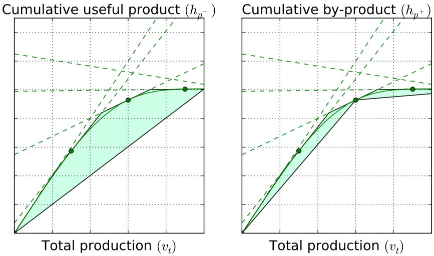

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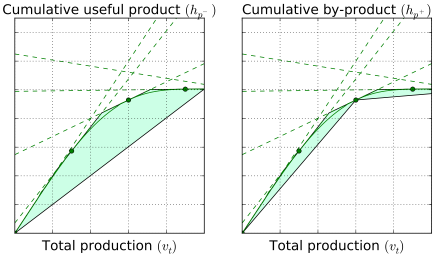

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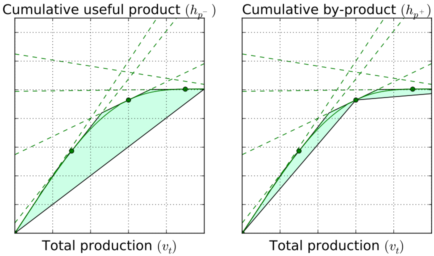

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Performance Evaluation

Experiments

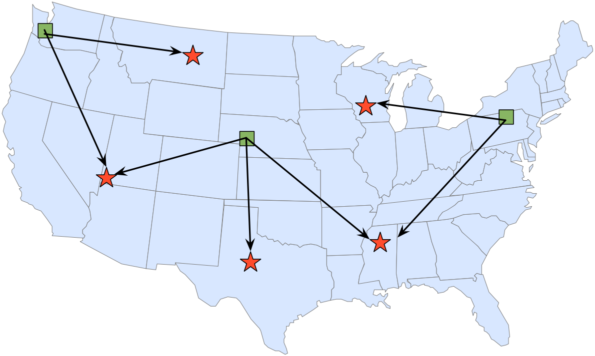
Goals

- Impact on formulation accuracy in going from F_1 to F_2
- Impact in solution time in going from F_1 to F_2 as solved by our models.

Sample Application

Transportation problem with production facilities manufacturing products for customers.

- Transportation problem with production facilities $\mathcal{I}$ manufacturing products $\mathcal{P}^+$ for customers $\mathcal{J}$.



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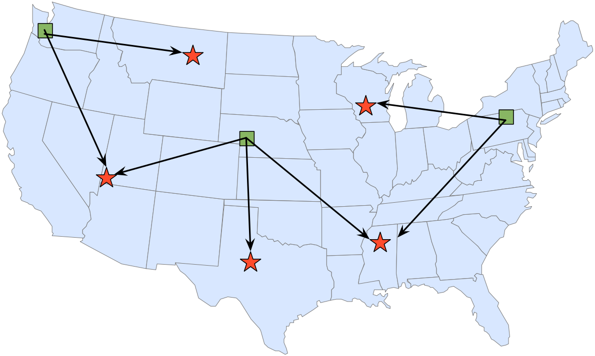

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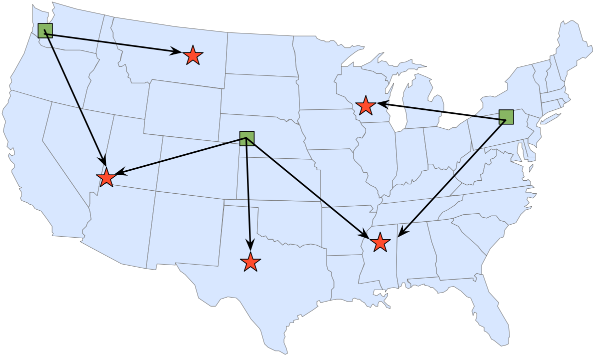

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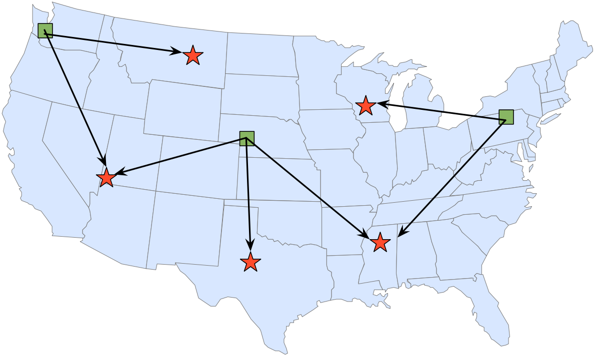

Comparing formulations: Small instances

Table: Comparing solution quality of the two different MINLP formulations F_1 and F_2 using BARON

$\lvert\mathcal{I}\rvert$	$\lvert\mathcal{T}\rvert$	$\lvert\mathcal{P}\rvert$	Formulation					Solution difference $\Delta y^*_{i,p,t}$ (*Range* : 0 — 30)	
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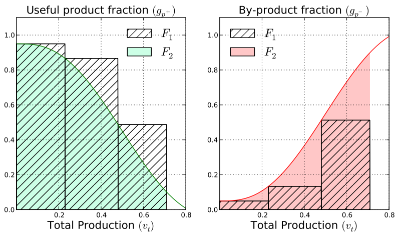
			Solution Bound	Best F_1 Feasible Solution	Repaired F_1 Solution	Solution Bound	Best F_2 Feasible Solution	Maximum $(\forall i, p, t)$	Average $(\forall i, p, t)$
5	5	2	0.171	0.200	0.272	0.208	0.219	5.17	0.47



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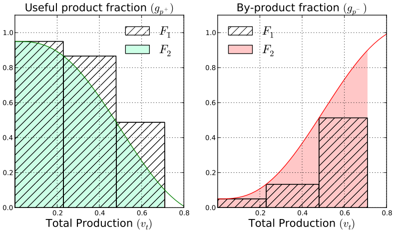

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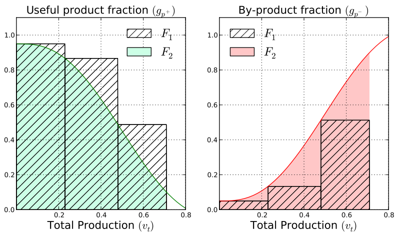

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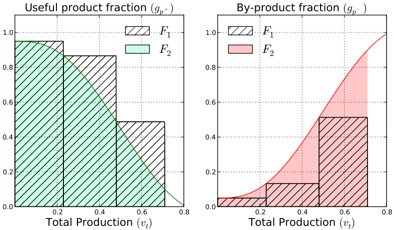

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5	5	2	0.157	0.175	0.243	0.190	0.198	4.68	0.40
5	10	2	0.255	0.369	0.381	0.325	0.340	0.41	0.06
5	10	2	0.256	0.358	0.388	0.324	0.341	1.33	0.12
5	10	2	0.303	0.377	0.464	0.385	0.399	3.14	0.34
10	10	2	0.357	0.607	0.770	0.637	0.670	4.49	0.32
10	10	2	0.507	0.784	0.954	0.797	0.820	3.84	0.32
10	10	2	0.377	0.692	0.754	0.645	0.675	2.60	0.13
15	10	2	0.656	1.085	1.308	1.100	1.141	3.84	0.30
15	10	2	0.540	0.960	1.053	0.903	0.945	2.16	0.14
15	10	2	0.552	1.033	1.090	0.901	0.940	1.01	0.08

Comparing MIP schemes: Large instances

Table: Comparing gaps of F_1 (with BARON) with MIP formulations (with Gurobi) of F_2 on large instances with more than 200 binary variables.

$\lvert\mathcal{I}\rvert$	$\lvert\mathcal{T}\rvert$	$\lvert\mathcal{P}\rvert$	Bounds (F_2)		Best F_2 feasible solution			Time (sec) / [Optimality gap (%)]			
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15	15	2	1394.13	1392.1	1412.07	1417.74	1416.98	[49.5]	[0.86]	[0.77]	[1.01]

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15	15	2	1394.13	1392.1	1412.07	1417.74	1416.98	[49.5]	[0.86]	[0.77]	[1.01]
15	15	4	1391.38	1385.82	1432.00	1431.74	1436.59	[50.2]	[1.60]	[1.41]	[1.62]
15	15	6	1283.03	1271.9	1326.2	1335.69	1330.13	[81.2]	[1.97]	[1.89]	[2.23]
15	20	2	1465.65	1465.4	1500.92	1510.79	1498.87	[53.0]	[1.90]	[1.67]	[1.72]
15	20	4	1573.95	1571.02	1663.04	1665.75	1691.03	[63.9]	[2.56]	[2.39]	[2.86]
15	20	6	1614.51	1608.73	1691.04	1691.4	1696.03	[83.1]	[3.12]	[2.71]	[3.09]
20	20	2	2185.07	2184.68	2245.19	2247.45	2254.25	[58.2]	[1.93]	[1.98]	[2.14]
20	20	2	1865.12	1863.33	1906.58	1906.93	1905.17	[49.1]	[1.24]	[1.46]	[1.57]
20	20	6	2058.69	2042.32	2163.22	2183.31	2185.59	-	[3.05]	[3.15]	[3.60]
25	25	2	3274.29	3270.23	3383.73	3381.22	3383.53	-	[2.28]	[2.35]	[2.63]
25	25	4	3222.66	3223.06	3417.42	3413.46	3437.34	[83.0]	[3.93]	[3.60]	[3.96]
25	25	6	2973.45	2963.5	4465.04	3919.11	4510.94	[83.2]	[32.2]	[22.9]	[33.6]

Comparing Algorithms: Large Instances

Table: Comparing gaps of F_1 (with BARON) with MIP formulations (with Gurobi) of F_2 on large instances with more than 200 binary variables.

$\|\mathcal{I}\|$	$\|\mathcal{T}\|$	$\|\mathcal{P}\|$	Bounds (F_2)		Best F_2 feasible solution			Time (sec) / [Optimality gap (%)]			
			1-SEC	k-SEC	PLA	1-SEC	k-SEC	F_1	PLA	1-SEC	k-SEC
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E. W. L. Beale and J. A. Tomlin.
Special facilities in a general mathematical programming system for non-convex problems using ordered sets of variables.
In J. Lawrence, editor, *Proceedings of the 5th International Conference on Operations Research*, pages 447–454, 1970.

Peter Gruber and Petar Kenderov.
Approximation of convex bodies by polytopes.
Rendiconti del Circolo Matematico di Palermo, 31:195–225, 1982.
10.1007/BF02844354.