Relaxations for Production Planning Problems with Increasing Byproducts

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Joint work with
Jeffrey Linderoth and James R. Luedtke

One slide summary

- **Problem Description**
  - Production process involves desirable & undesirable products.

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- **Performance evaluation**

Problem Description

Production Process

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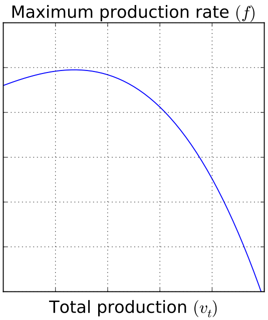
- Decisions span a planning horizon $\mathcal{T}$.
- Discrete decisions determine the start time of the production process.
- Continuous decisions determine the production profile evaluated by production functions $f(\cdot)$ and $g_p(\cdot)$.

Production functions

- Production function $f(\cdot)$ is a concave function that determines the maximum production rate as a function of total production.



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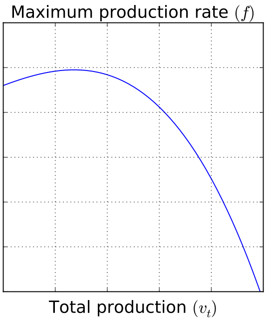

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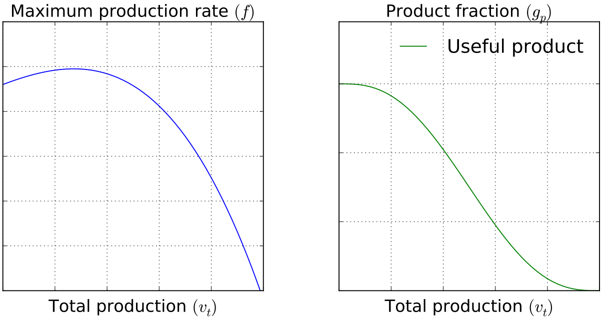

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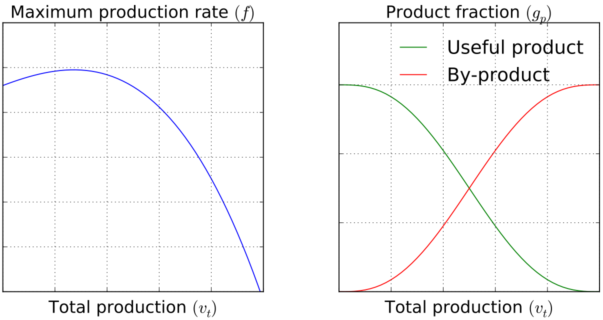

Continous time formulation

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Production profiles are active only after the start time $z(t)$

$$v(t) \leq \mathrm{M}\ z(t)$$

Discrete Time Formulations

Discrete time formulations

Past models have proposed a natural discretization of this continuous time model.

Continuous time formulation (CNT)

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z_t	Facility on/off decision variable.

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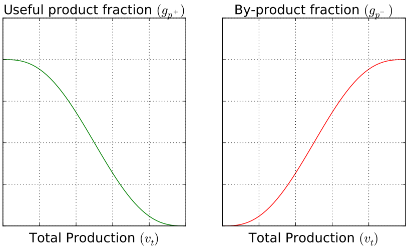

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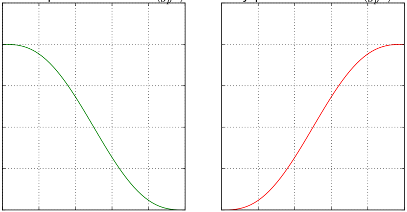

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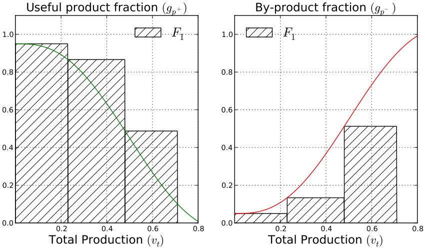

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$$\stackrel{\text{def}}{=} h_p(v_t)$$

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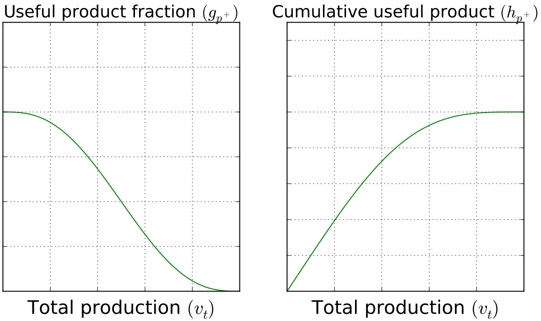
Key Idea

- Integral of a non-increasing function is concave.



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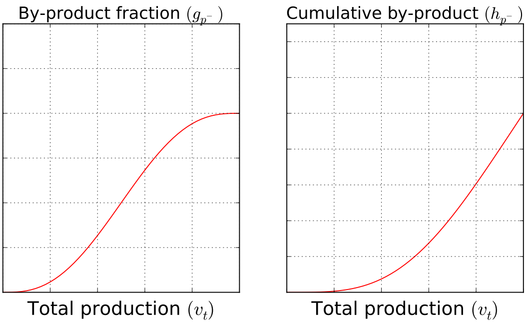

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- Integral of a non-increasing function is concave.
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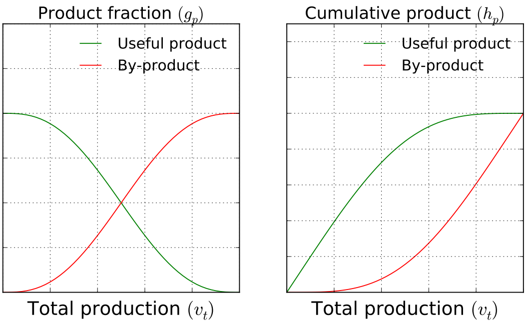
- Lets deal with h_p instead of g_p!



Comparing formulations

What have we done so far ?

Formulation F_1

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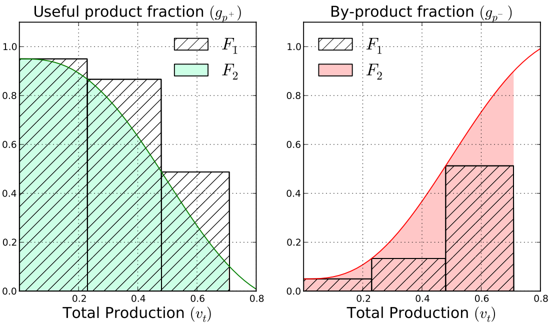

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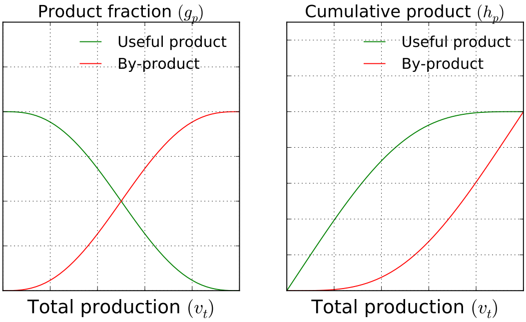
- F_2 is computationally better because it deals with convex functions while F_1 deals with bilinear terms.



MIP Approximation & Relaxations

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Approximations & Relaxations

Mixed Integer Non-Linear Programs (MINLP)

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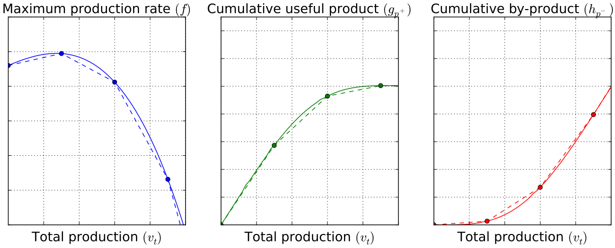
But...there is hope

We only need to approximate or relax univariate convex and concave functions.

Approximations & Relaxations I

Piecewise Linear Approximation (PLA)

Approximate all the nonlinear production functions with piecewise linearizations.



Approximations & Relaxations I

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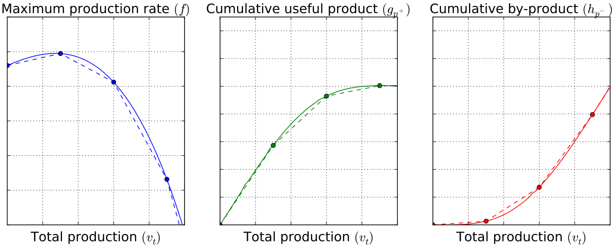

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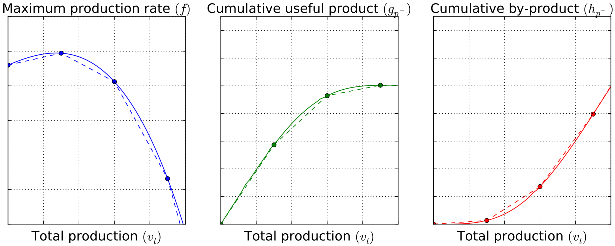

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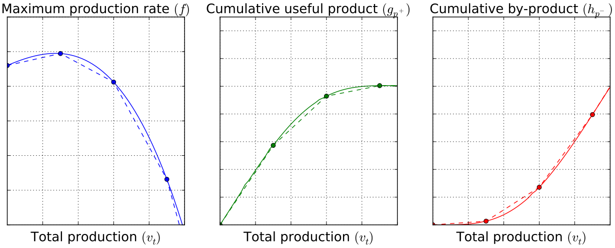
  - NOT a relaxation of the original formulation.



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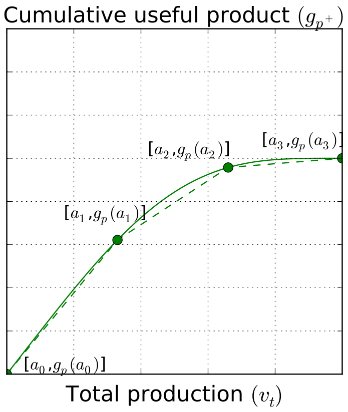
Specially ordered sets (SOS)



Approximating $g_p(v_t)$

$$g_p(v_t) \approx \sum_{o \in \mathcal{O}} \lambda_{t,o} g_p(a_o)$$

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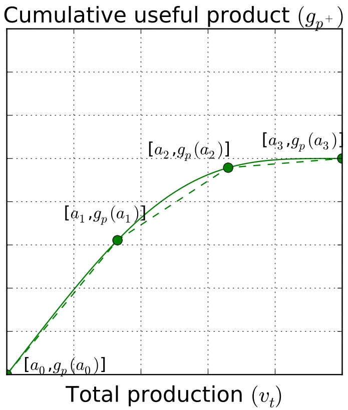

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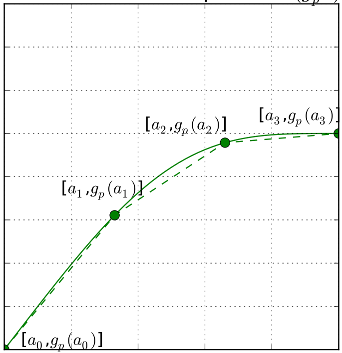

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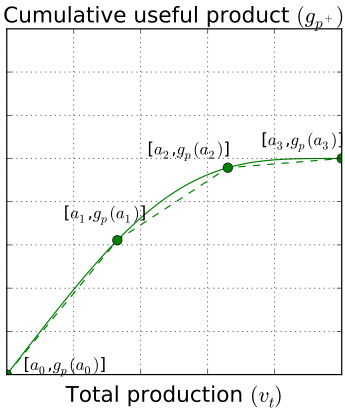

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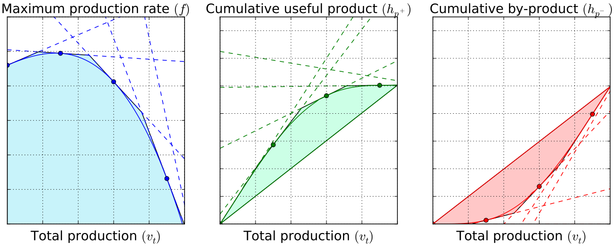
Relax all the nonlinear production functions using inner and outer approximations.



Approximations & Relaxations II

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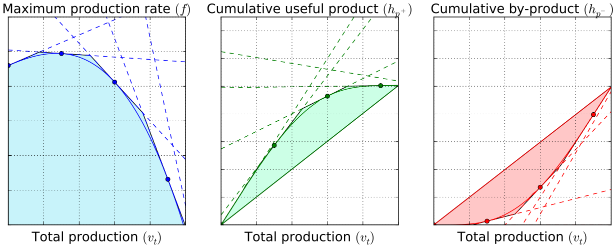
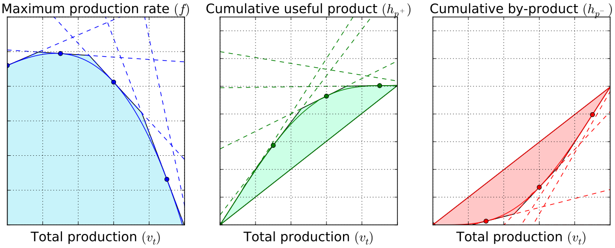


Approximations & Relaxations II

Secant Relaxation (1-SEC)

Relax all the nonlinear production functions using inner and outer approximations.

- Pros
  - Relaxation of the original formulation.
  - Does NOT introduce additional SOS2 variables.
- Cons
  - May not be ‘close’ to a feasible solution of the MINLP formulation.



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Approximations & Relaxations III

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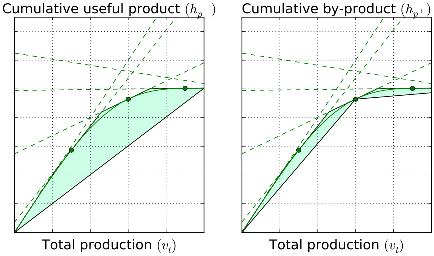
Relax all the nonlinear production functions using inner and outer approximations but use multiple secants instead of a just a single one.



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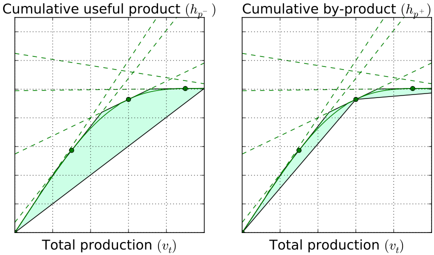

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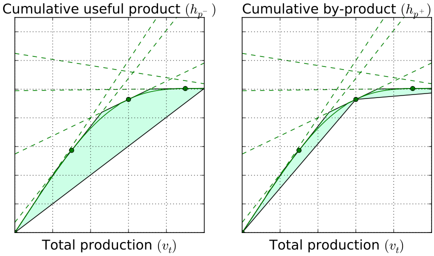

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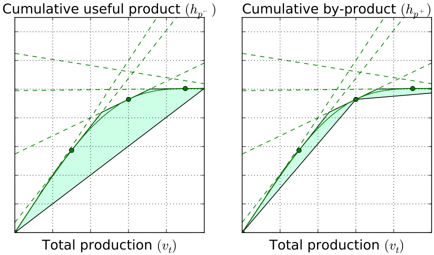

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- Production functions are positive only if the facility is open.
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Trix

SOS2/Hull binary trick

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Stronger Formulation...

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Performance Evaluation

Experiments

Goals

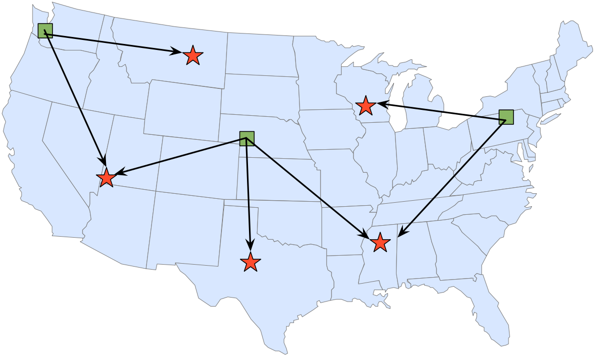
- Impact on formulation accuracy in going from F_1 to F_2
- Impact in solution time in going from F_1 to F_2 as solved by our models.
- Impact of stronger formulations on solving the MIP approximation/relaxations.

Sample Application

Transportation problem with production facilities manufacturing products for customers.

Performance Evaluation

- Transportation problem with production facilities $\mathcal{I}$ manufacturing products $\mathcal{P}^+$ for customers $\mathcal{J}$.



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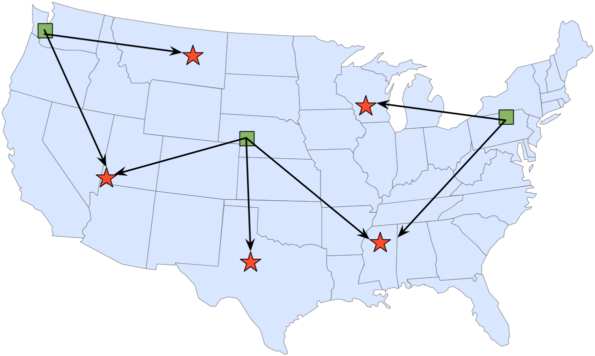

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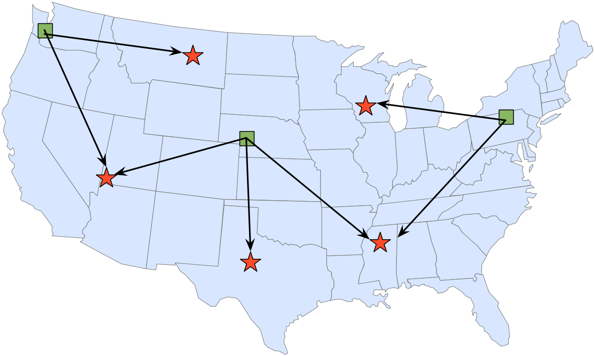

Performance Evaluation

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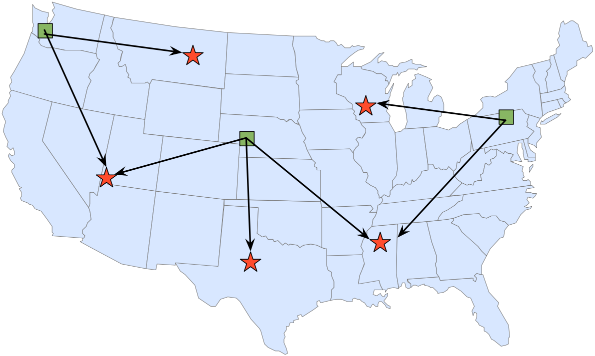
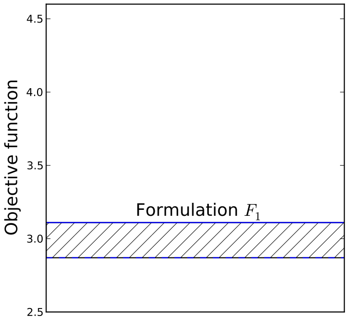
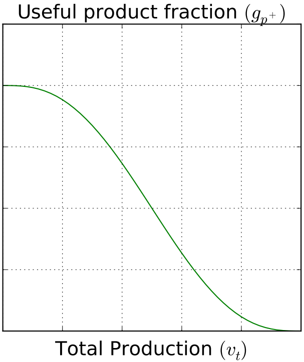


Accuracy

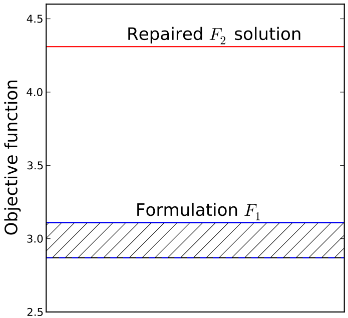


Objective function
4.5
4.0
3.5
3.0
2.5
Formulation F_1



Useful product fraction (g_{p^+})
Total Production (v_t)

Accuracy


Repaired F_2 solution
Formulation F_1
Objective function
4.5
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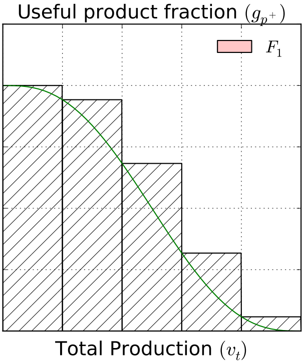
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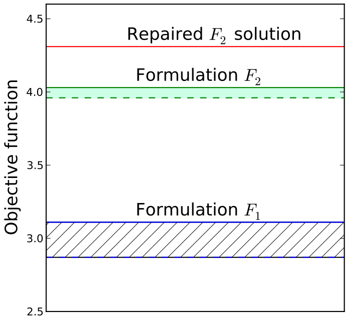
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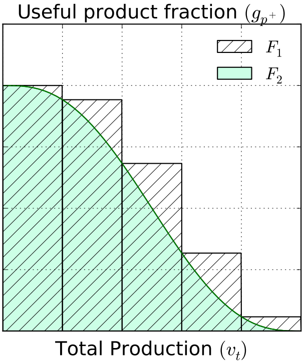
Useful product fraction (g_{p^+})
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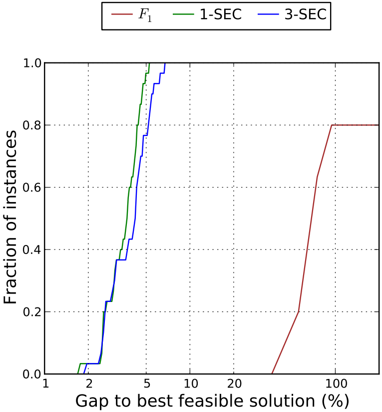
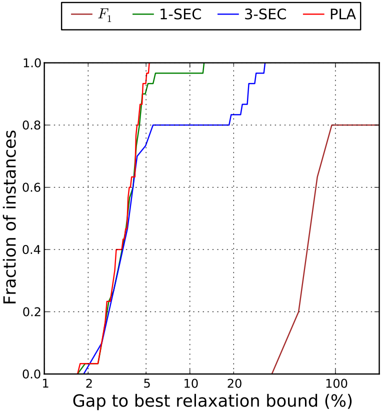
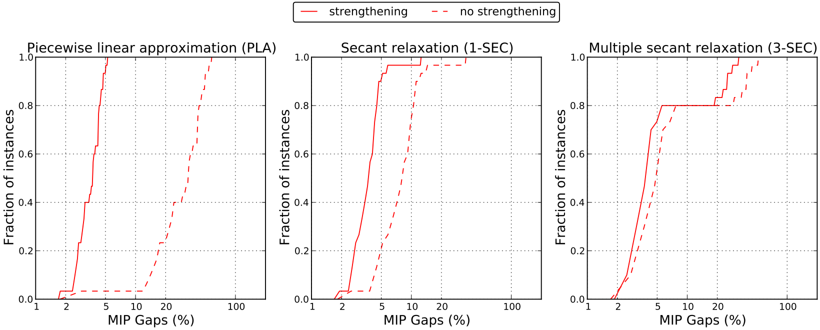
Formulations





Trix



strengthening
no strengthening
Piecewise linear approximation (PLA)
Secant relaxation (1-SEC)
Multiple secant relaxation (3-SEC)
Fraction of instances
MIP Gaps (%)
0.0
0.2
0.4
0.6
0.8
1.0
1
2
5
10
20
100

Conclusions

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  - Devised scalable MIP approximations & relaxations (PLA, 1-SEC, k-SEC).

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  - Defined a non-convex production process involving desirable & undesirable products.
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Thats all folks!