Relaxations for Production Planning Problems with Increasing Byproducts

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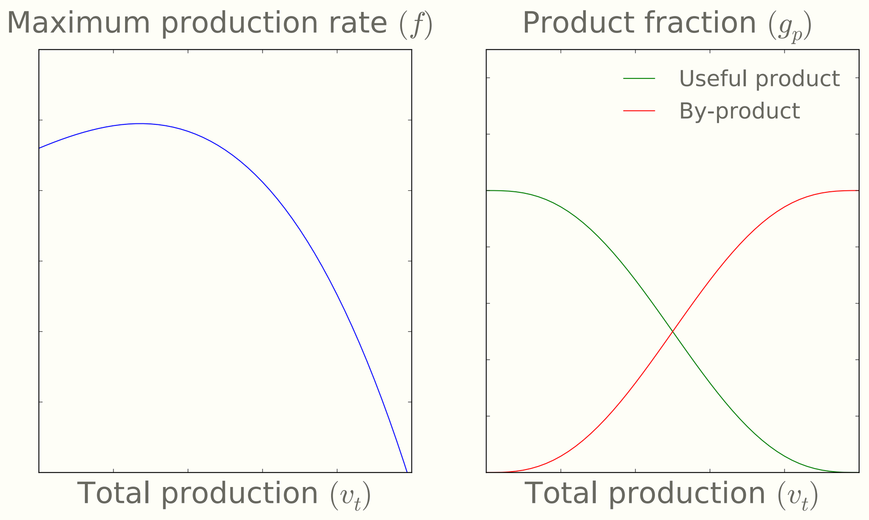
background

We study a **production** planning problem, where the production process creates a set of **products** ($\mathcal{P} = \mathcal{P}^+ \cap \mathcal{P}^-$) , a subset of which are **useful** ($\mathcal{P}^+$) with the remaining undesirable **byproducts** ($\mathcal{P}^-$).



★ Fraction of **useful products/byproducts** monotonically **decreases/increases** as a function of total production.

Problems with these characteristics arise in applications like natural resource extraction, hydro turbine performance modeling and compressor scheduling in petroleum reservoirs.

problem statement

Continuous time formulation

$$\mathbf{v(t)} = \int_0^{\mathbf{t}} \mathbf{x(s)ds} \qquad \forall t \in [0, T]$$

$$\mathbf{x(t)} \leq \mathbf{f(v(t))} \qquad \forall t \in [0, T]$$

$$\mathbf{y_p(t)} = \mathbf{x(t)\ g_p(v(t))} \qquad \forall p \in \mathcal{P}^+, t \in [0, T]$$

$$\mathbf{v(t)} \leq \mathrm{M}\mathbf{z(t)} \qquad \forall t \in [0, T]$$

Decision variables

- $x(t)$ mixture production rate at time t
- $v(t)$ cumulative production
- $y_p(t)$ product production
- $z(t)$ facility on/off

Amount of each product produced is a **non-convex** function of the cumulative production up to that time instance!

discrete time

★ **Key Idea:** Integral of monotonically increasing/decreasing functions are convex/concave.

Formulation X_1

$$v_t = \sum_{s \leq t} x_s$$

$$x_t \leq \Delta_t f(v_{t-1})$$

$$y_{p,t} = x_t g_p(v_{t-1})$$

$$v_t \leq \mathrm{M} z_t$$

$$z_t \leq z_{t+1}$$

Tarhan et. al.

Formulation X_2

$$v_t = \sum_{s \leq t} x_s$$

$$x_t \leq \Delta_t f(v_{t-1})$$

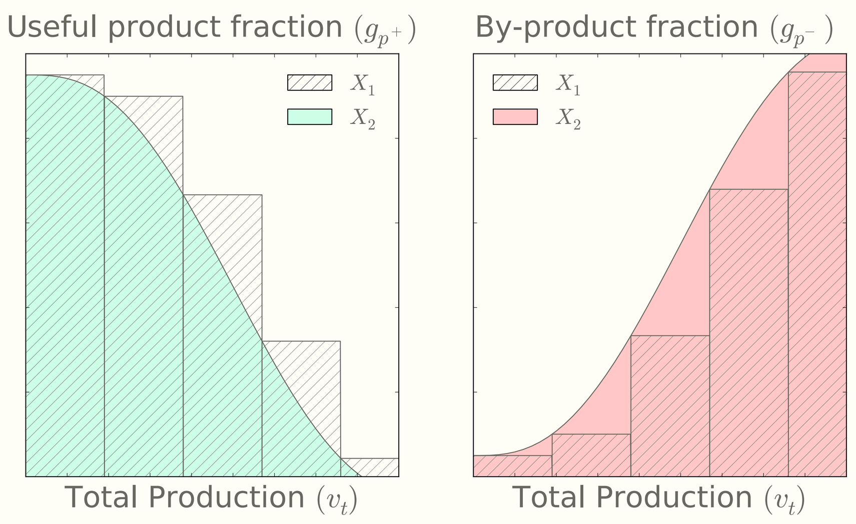
$$w_{p,t} = \int_0^{v_t} g_p(\theta) d\theta \stackrel{\text{def}}{=} h_p(v_t)$$

$$y_{p,t} = w_{p,t} - w_{p,t-1}$$

$$v_t \leq \mathrm{M} z_t$$

$$z_t \leq z_{t+1}$$

comparing formulations



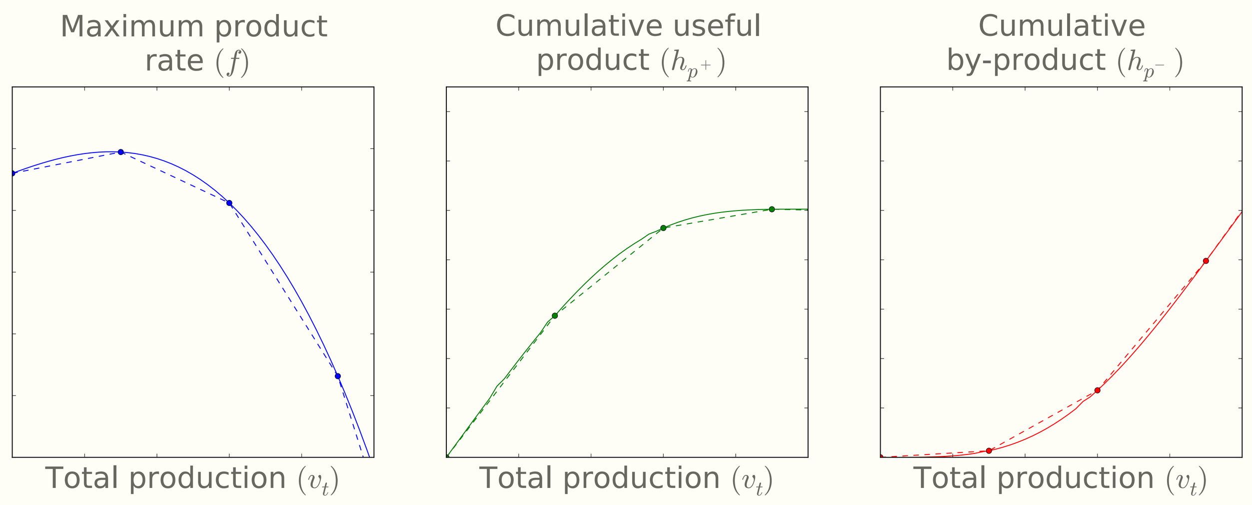
★ X_2 is more **accurate**

★ X_2 is computationally more efficient because it only requires the approximation of **univariate convex/concave** functions

mip formulations

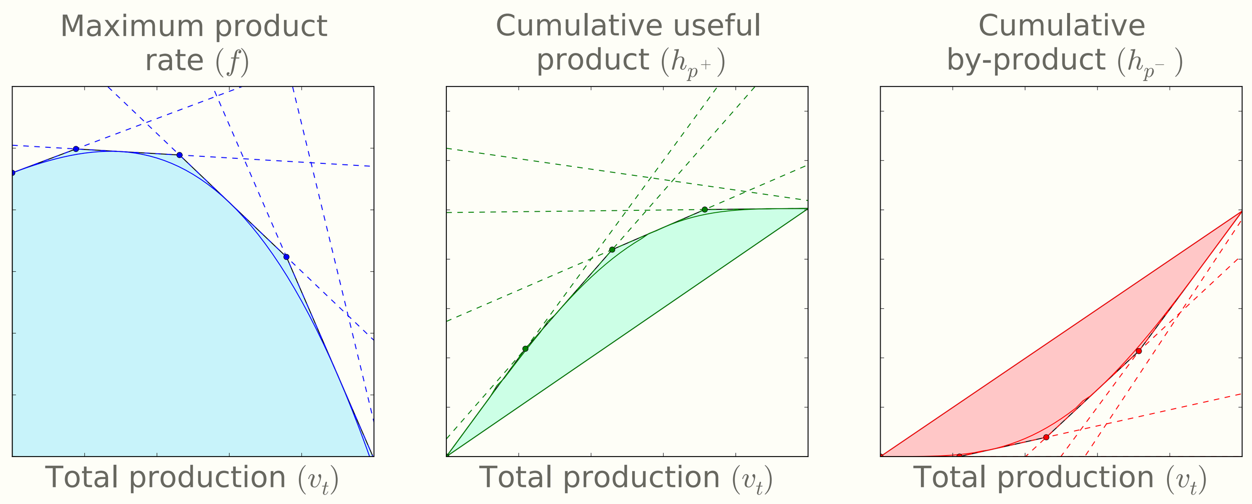
We propose three different MIP formulations for X2. The functions share the domain $\mathbf{v_t}$ and can be written as a convex combination of the breakpoints $\{\mathbf{B_o} \mid \forall \mathbf{o} \in \mathcal{O}\}$ using variables $\{\lambda_\mathbf{o} \mid \forall \mathbf{o} \in \mathcal{O}\}$.

Piecewise Linear Approximation (PLA)



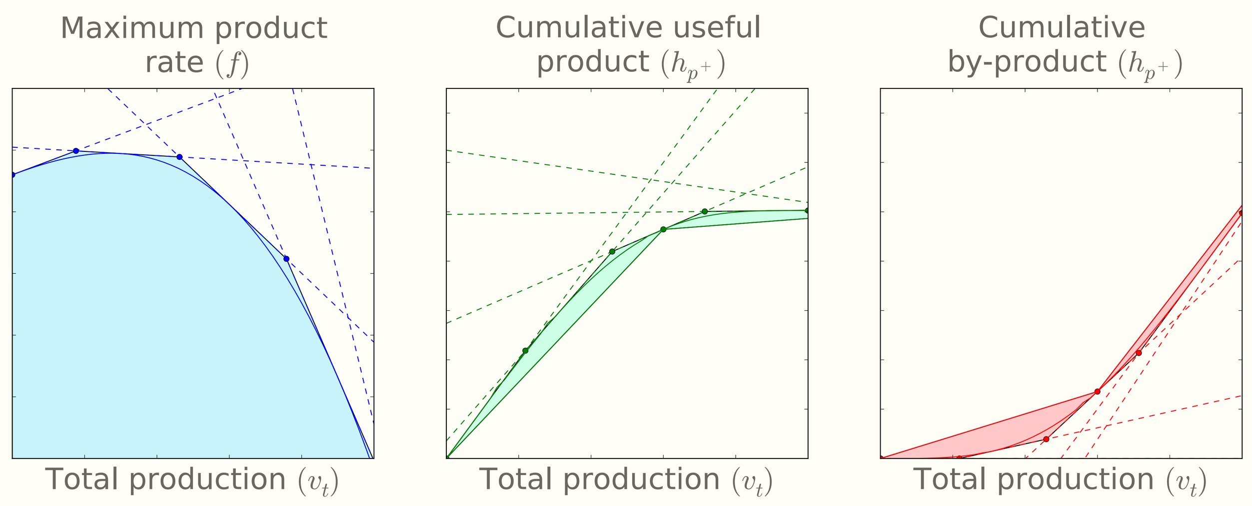
$$\mathbf{x_t} \leq \mathbf{\Delta_t} \sum_{\mathbf{o} \in \mathcal{O}} \mathbf{F_o} \lambda_{\mathbf{t,o}} \qquad \mathbf{w_{p,t}} = \sum_{\mathbf{o} \in \mathcal{O}} \mathbf{H_{p,o}} \lambda_{\mathbf{t,o}} \qquad \{\lambda_{\mathbf{t,o}} \mid \forall \mathbf{o} \in \mathcal{O}\} \in \mathrm{SOS2}$$

1-Secant Relaxation (1-SEC)



$$\mathbf{x_t} \leq \mathbf{\Delta_t} \sum_{\mathbf{o} \in \mathcal{O}} \hat{\mathbf{F}}_\mathbf{o} \lambda_{\mathbf{t,o}} \qquad \mathbf{w_{p,t}} = \sum_{\mathbf{o} \in \mathcal{O}} \hat{\mathbf{H}}_{\mathbf{p,o}} \lambda_{\mathbf{t,o}}$$

k-Secant Relaxation (k-SEC)



$$\mathbf{x_t} \leq \mathbf{\Delta_t} \sum_{\mathbf{o} \in \mathcal{O}} \hat{\mathbf{F}}_\mathbf{o} \lambda_{\mathbf{t,o}} \qquad \sum_{\mathbf{o} \in \mathcal{O}} \hat{\mathbf{H}}_{\mathbf{p,o}} \lambda_{\mathbf{t,o}} \leq \mathbf{w_{p,t}} \leq \sum_{\mathbf{o} \in \mathcal{O}} \mathbf{H_{p,o}} \lambda_{\mathbf{t,o}}$$

$$\{\lambda_{\mathbf{t,o}} \mid \forall \mathbf{o} \in \mathcal{O}\} \in \mathrm{SOS2}$$

strengthening

In all MIP formulations, the production functions are positive only if the facility is open. We leverage this property to tighten all formulations using the following trick!

$$\mathbf{z_t} = \sum_{\mathbf{o} \in \mathcal{O}} \lambda_{\mathbf{t,o}}$$

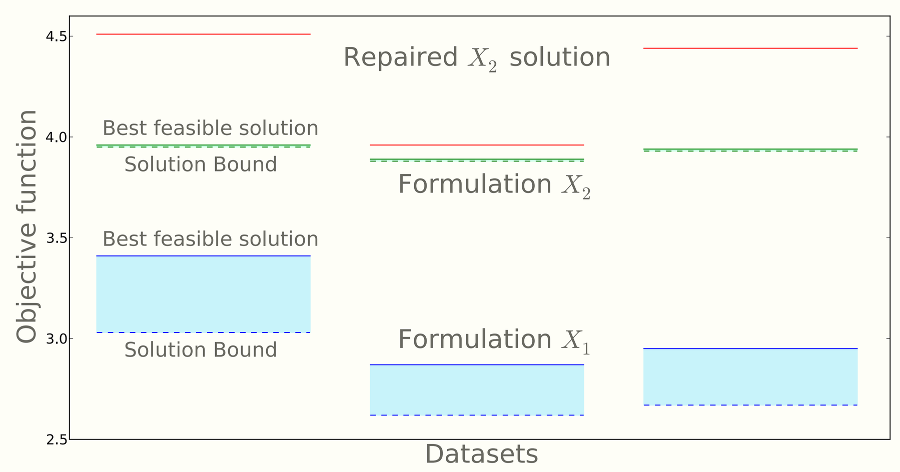
★ No need for **variable upper bounds**

★ **Locally ideal!**

results

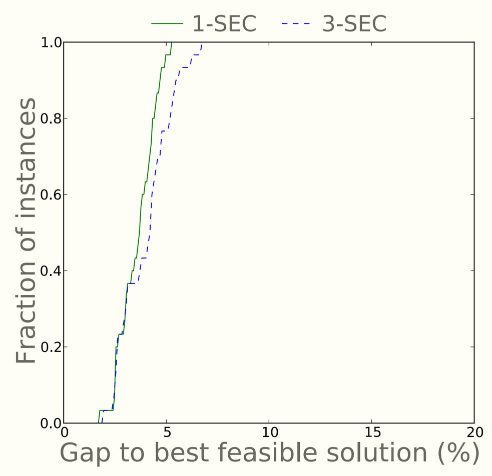
We conducted numerical experiments on 60 datasets of a production planning problem.

Accuracy of X_1 vs X_2



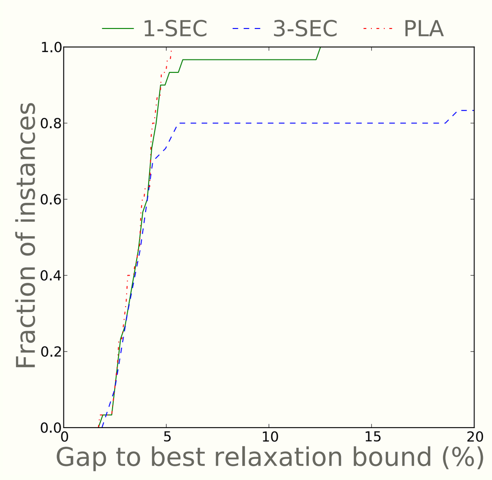
Inaccuracy of formulation X_1 leads to significantly worse solutions!

Comparing MIP Formulations



Quality of **solution bounds**

Method	Gap to best feasible solution (%)		
	A.M	Variance	G.M
X_1	58.96	57.36	13.44
1-SEC	3.59	3.47	0.89
3-SEC	3.99	3.80	1.25

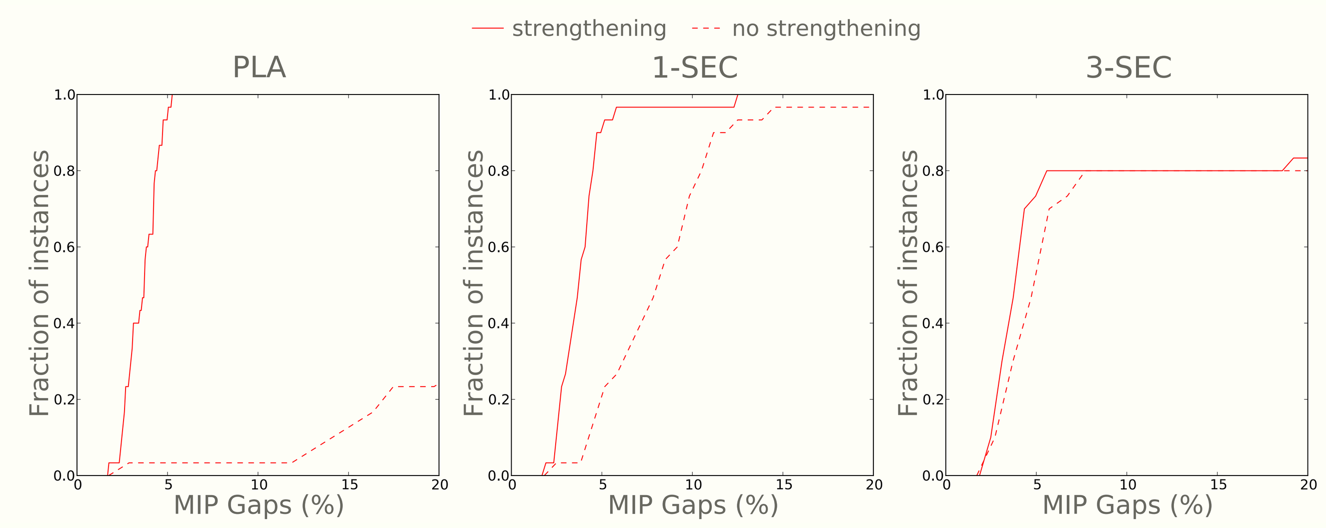


Quality of **feasible solutions**

Method	Gap to best bound (%)		
	A.M	Variance	G.M
X_1	58.96	57.36	13.44
PLA	3.59	3.47	0.90
1-SEC	3.90	3.63	1.86
3-SEC	7.90	5.04	9.19

1-SEC yields best bounds and competitive feasible solutions.

Effect of strengthening



Strengthening greatly improves all MIP formulations.